PROPERTIES OF WIGNER DISTRIBUTION FUNCTIONS APPLIED TO QUANTUM MECHANICS

Alexander Gorbachev, PFUR
Wigner distribution functions

First introduced by Eugene Wigner in 1932

\[ W_\varphi(q, p) = \frac{1}{\pi \hbar} \int \varphi^*(q + q') \varphi(q - q') e^{\frac{2ipq'}{\hbar}} dq' \]

Idea: Link the wave function to a probability distribution in phase space

Application: Evaluation of expectation values of Hermitian operators

\[ \langle A \rangle_\rho = \int A(q, p) W_\rho(q, p) dq dp \]

Disadvantage: Wigner function is a quasi-probability distribution with possible "negative probabilities"
Alternative

We can use operational quantum distribution function instead

\[ P(q, p) = W_{\rho_1} \ast W_{\rho_2}(p, q) \]

States of the quantum object and quantum filter before the measurement procedure

distribution of measured values of the observed composite system

Now the average value of the measured observable is equal to

\[ \langle A \rangle_{\rho_1 \otimes \rho_2} = \int A(q, p) (W_{\rho_1} \ast W_{\rho_2})(q, p) dq dp \]

Advantage: Convolution of any two Wigner quantum distribution functions is a positive definite probability distribution
Formulation of the problem

On the other hand, the average value of the measured observable is equal to
\[
\langle A \rangle_{\rho_1 \otimes \rho_2} = Tr \left\{ \left( O_{w} (A) \otimes \hat{I} \right) (\hat{\rho}_1 \otimes \hat{\rho}_2) \right\} = Tr_1 \left\{ O_{\rho_2} (A) \hat{\rho}_1 \right\}
\]

As a result, the following theorem is valid:
Quantization rule of Kuryshkin-Wodkiewicz corresponds a continuous linear operator of the form \( O_{\rho_2} (A) = O_{w} (A * W_{\rho_2}) : S(Q) \rightarrow S'(Q) \) to the distribution \( A \in S'(T * Q) \)

To prove this we use the properties of the Wigner quantum distribution functions \( W(-q, p) = -W(q, p) \) and \( W_{\varphi} (q, p) = W_{\tilde{\varphi}} (p, q) \)
Proof scheme

Three common forms of Fourier transform:

\[ \hat{\phi}(y) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} \phi(x) e^{-iyx/\hbar} dx \]

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\[ \hat{\phi}(y) = \int_{-\infty}^{+\infty} \phi(x) e^{-\pi yx} dx \]

\[ W(\varphi, q, p) = 2\pi\hbar \cdot W_{\hat{\varphi}}(p, q) \]

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\[ W(-q, p) = -W(q, p) \]

\[ W(q, -p) = -W(q, p) \]

\[ W(-q, -p) = W(q, p) \]
Multidimensional case

n-dimensional Wigner distribution functions

\[ W_\phi(q, p) = (\pi \hbar)^{-n} \int \ldots \int \phi^* (q_1 + q'_1 + \ldots + q_n + q'_n) \phi(q_1 - q'_1 + \ldots + q_n - q'_n) e^{\frac{2i(p_1q'_1 + \ldots + p_nq'_n)}{\hbar}} dq'_1 \ldots dq'_n. \]

\[ \hat{\phi}(\bar{y}) = \frac{1}{(2\pi \hbar)^{n/2}} \int_{\mathbb{R}^n} \phi(\bar{x}) e^{-iy\bar{x}/\hbar} d\bar{x} \]
\[ \hat{\phi}(\bar{y}) = \int_{\mathbb{R}^n} \phi(\bar{x}) e^{-iy\bar{x}} d\bar{x} \]
\[ \hat{\phi}(\bar{y}) = \int_{\mathbb{R}^n} \phi(\bar{x}) e^{-2\pi iy\bar{x}} d\bar{x} \]

\[ W_\phi(q, p) = (2\pi \hbar)^n \cdot W_\phi(p, q) \]
\[ W_\phi(q, p) = W_\phi(p, q) \]
\[ W_\phi(q, p) = W_\phi(p, q) \]

\[ W(-q, p) = (-1)^n W(q, p) \]
\[ W(q, -p) = (-1)^n W(q, p) \]
\[ W(-q, -p) = W(q, p) \]
Results

The presence of a complete system of almost orthogonal (Sturmian) functions allows to realize the stable numerical method for investigating the discrete spectrum of the measured observables of an open quantum system.
Thank you!